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Physics & Number Theory

Mathematics plays a major role in physics. Newton's law of motion relating force to acceleration requires students to quickly learn how to solve ordinary differential equations. Fourier invented his famous series to solve heat flow problems. Vector calculus is needed to understand electromagnetism, fluid flow etc. Complex numbers naturally occur when considering oscillatory systems. Tensors, and non-Euclidean geometry are needed in the study of general relativity. More recently, the realisation that symmetry principles underly many important physical theories has led to the growing use of group theory in physics.

However, one important area of mathematics, often seen as somewhat esoteric, that is rarely taught to physics students, is number theory. I think one of the main reasons this subject is not taught is due to the outstanding success of calculus in physical theories, and basic calculus assumes continuous functions and continuous variables. Discrete variables, such as integers, do not sit well with calculus, and so this whole body of mathematics is often conveniently left out of a physicist's education. Number theory mainly concerns itself with relationships between natural numbers, integers (and proper fractions). There are many well known problems in number theory, a large number of which are still conjectures. Is number theory relevant to physics – or turning this around, can physics give any insights into number theory ? I would argue that there are many examples in physics where integers (or fractions) occur, and there are some tantalising clues to link physics with number theory.

One well-known example where integers occur in physics is in Millikan's famous oil drop experiment to measure the charge on an electron – in this case Millikan found charge came in integral quantities, the smallest of which he inferred to be the charge on a single electron. In this case, charge is not a continuous variable, but a discrete variable, only coming in multiples of the basic unit.

Planck resolved the "ultraviolet catastrophe" of blackbody radiation by arguing that electromagnetic modes could only have energies that were multiples of $h\nu$ (where h is Planck's constant, and ν is the frequency), a result used by Einstein to explain the photoelectric effect. Again, in this instance, energy is not a continuous variable but comes in discrete chunks.

Integer sequences may be generated naturally in physics by considering eigenvalue/eigenfunction problems. Early on in a quantum mechanics course, students calculate the energy eigenvalues of a particle in a one-dimensional box (where the potential inside the box is zero, and is infinite outside the box). The energy eigenvalues, normalised to the ground state value, form the sequence of square numbers – 1,4,9,16,25…. This may seem to be rather a complex way of generating such a simple integer sequence, but this approach has the advantage of further generalisation. A more advanced quantum mechanics student may well consider the problem of an infinitely deep one-dimensional parabolic well – in this case, the energy eigenvalues, again suitably normalised, form a uniform sequence – 1,2,3,4,5….Is it possible that any sequence of integers can be generated by using an appropriate potential function in the one-dimensional Schrödinger equation ? The answer appears to be yes (provided the numbers in the sequence increase at a slower rate than n^2), and it has been suggested that a suitable potential function may be

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capable of generating the sequence of prime numbers. It is here that some physical insight may be useful. It is conjectured (but not yet proved) that there are infinitely many primes that differ just by 2. In quantum mechanics we know that energy levels may be split by application of a magnetic field – could such a term in the potential function help to explain why some of the prime numbers differ by 2 ?

The physics of musical instruments, whether they be of the stringed or woodwind variety can also lead to integer eigenvalues (i.e. the relative frequencies of the notes which are produced), although in practice the sequence of notes produced will not stretch to infinity !

Aside from integers, fractional values can also occur. Examples here include the Hydrogen atom, where the energy eigenvalues vary as $1/n^2$ (where n is an integer) which leads to the observed Balmer sequence of spectral lines (where spectral lines appear at frequencies of the form: $1/n^2 - 1/m^2$).

Looking towards the future, there is increasing speculation that both time and space may be discrete (at the Planck scale). If this is found to be the case, integer multiples of these basic quantities could better explain Planck-scale physics than would conventional continuous variables. Perhaps the next generation of physicist should learn more about number theory to get a better understanding of quantum gravity !

This would bring us around almost full circle. It is not perhaps widely known that Riemann, one of the greatest number theorists of all time, also worked on physics problems. On the same piece of paper where he composed the Riemann hypothesis (concerned with the distribution of zeros of the Riemann zeta function – important for the distribution of prime numbers) he also studied the conditions under which the oscillations of a gravitationally bound rotating sphere of fluid would be stable – in this case the physical problem studied bore some relation to the number theory problem he was working on, since for stability all the eigenvalues had to lie on a straight line. Recently, computer investigations into the spacings of the location of the zeros of Riemann's zeta function have shown that they are distributed in the same way as the energy eigenvalues of a particular quantum chaotic system. It is not completely out of the question that some of the most pressing problems in number theory, which have been around for centuries, could in the near future be solved by physicists !

Further Reading

1. John Derbyshire, "Prime Obsession", Joseph Henry Press, 2002
2. http://www.maths.ex.ac.uk/~mwatkins/zeta/physics.htm
3. G. Mussardo, "The Quantum Mechanical Potential for the Prime Numbers", arXiv: cond-mat/9712010
4. Marcus du Sautoy, "The Music of the Primes", Fourth Estate, 2003

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